MATH 3235 Probability Theory

9/19/22

Last Time:

Expectation and Variance

of several example of r.v.

Conditional Expectation.

If $\mathbb{P}$ is a probability and

B is an event with $\mathbb{P}(B) > 0$

Then

$\mathbb{P}(\cdot \mid B)$ is a probability.

If X is a r.v. we can

define

$$p(x \mid B) = \mathbb{P}(X = x \mid B)$$

$p(\cdot \mid B)$ is a p.m.f.

$$\mathbb{E}(X \mid B) = \sum_x x \, p(x \mid B) =$$

$$= \sum_x x \, \mathbb{P}(X = x \mid B)$$

Conditional expectation of X given B.

Th: If B_i, $i = 1, 2, \ldots$ form a partition of Ω with $\mathbb{P}(B_i) > 0$ for every i Then

$$\mathbb{E}(X) = \sum_i \mathbb{E}(X \mid B_i) \mathbb{P}(B_i)$$

Proof:

$$\sum_i \mathbb{E}(X \mid B_i) \mathbb{P}(B_i) =$$

$$\sum_i \sum_x x \, \mathbb{P}(X = x \mid B_i) \mathbb{P}(B_i) =$$

$$\sum_x x \sum_i \mathbb{P}(X = x \mid B_i) \mathbb{P}(B_i) =$$

$$\sum_x x \, \mathbb{P}(X = x) = \mathbb{E}(X).$$

Flip a coin with prob. of H equal To p. How long is The initial sequence of equal flips?

X

HHHT $\qquad X = 3$

TTTTH $\qquad X = 4$

$$\mathbb{E}(X \mid \text{first flip is } H) = \frac{1}{q}$$

$$\mathbb{E}(X \mid \text{first flip is } T) = \frac{1}{p}$$

$$\mathbb{E}(X) = \mathbb{E}(X \mid H)\,\mathbb{P}(H) + \mathbb{E}(X \mid T)\,\mathbb{P}(T) =$$

$$= \frac{p}{q} + \frac{q}{p} = \frac{1}{pq} - 2$$

if $p = \frac{1}{2} \Rightarrow \mathbb{E}(X) = 2$

Multivariate discrete distributions.

You have 2 dice. Roll them.

They are different.

$$\Omega = \{(i,j) \mid 1 \leq i,j \leq 6\}$$

$$\omega = (i,j)$$

$$X(\omega) = i \qquad Y(\omega) = j$$

$$\mathbb{P}(\{(i,j)\}) = \frac{1}{36}$$

Joint p.m.f. of X and Y

$$p(x,y) = \mathbb{P}(X = x \ \& \ Y = y)$$

	1	2	3	4	5	6
1	1/36	1/36	. . . .			
2	1/36	1/36	. . . .			
3	⋮	⋮	⋱			
4	⋮	⋮				
⋮						

X and Y are called jointly distributed r.v.

$p(x,y)$ is The joint p.m.f.

Education and Income

X education 0 1 2
h.s g pg

Y income 0 1 2
low med. high

	0	1	2
0	0.2	0.1	0
1	0.1	0.3	0.1
2	0	0.1	0.1

$p(x,y)$ according To The Table.

Properties

$$p(x,y) \geq 0$$

$$\sum_{x,y} p(x,y) = 1$$

What if I only care for one of the two? Let's say X.

$$p_X(x) = \mathbb{P}(X = x) =$$

$$= \sum_y \mathbb{P}(X = x \;\&\; Y = y) =$$

$$= \sum_y p(x,y)$$

Example 1 (dice)

$$p_X(x) = \frac{1}{6} \qquad x = 1, 2 \ldots 6$$

$$p_Y(y) = \frac{1}{6} \qquad y = 1, 2 \ldots 6$$

Example 2: sample

			X		
		0	1	2	p_Y
Y	0	0.2	0.1	0	0.3
	1	0.1	0.3	0.1	0.5
	2	0	0.1	0.1	0.2
	p_X	0.3	0.5	0.2	

$$p_X(0) = p(0,0) + p(0,1) + p(0,2) = 0.3$$

$$p_X(x) = \sum_y p(x,y)$$

Marginal of $p(x,y)$ over x

x-marginal of $p(x,y)$

$$p_Y(y) = \sum_x p(x,y)$$

y-marginal of $p(x,y)$

Expected values.

$$f(x,y) = x+y$$

$$g(x,y) = \begin{cases} 1 & x>y \\ 0 & \text{otherwise} \end{cases}$$

$$Z = f(X,Y)$$

$\omega = (i,j)$ $\qquad Z(\omega) = i+j.$

$$\mathbb{P}(Z=z) = \sum_{\substack{(x,y) \\ f(x,y)=z}} p(x,y)$$

$$\mathbb{P}(Z=2) = p_Z(2) = \frac{1}{36}$$

$$\mathbb{P}(Z=3) = p_Z(3) = \frac{1}{18}$$

$$p_Z(4) = \frac{1}{12}$$

$\vdots$

If X and Y are discrete r.v.

$f : \mathbb{R}^2 \to \mathbb{R}$

$Z = f(X, Y)$ is a discrete r.v.

$$p_Z(z) = \sum_{(x,y) \in f^{-1}(z)} p(x, y)$$

Thm:

$$\mathbb{E}(f(X, Y)) = \sum_{x,y} f(x,y)\, p(x,y)$$

Cor: if X and Y are r.v. with j. p. m. f. $p(x,y)$ and a, b are real numbers Then:

$$\mathbb{E}(aX + bY) = a\,\mathbb{E}(X) + b\,\mathbb{E}(Y)$$

$$\mathbb{E}(X) = \sum_{x} x \, p_X(x) = \sum_{x,y} x \, p(x,y)$$

$$\mathbb{E}(Y) = \sum_{y} y \, p_Y(y) = \sum_{x,y} y \, p(x,y)$$

$$a\mathbb{E}(X) + b\mathbb{E}(Y) =$$

$$\sum_{x,y} (ax + by) \, p(x,y) =$$

$$\mathbb{E}(aX + bY).$$

Are they independent?

X and Y are independent iff for every x and y

$$p(x,y) = p_X(x)\, p_Y(y)$$

This means that

The events $\{X = x\}$ and $\{Y = y\}$ are independent

Remark: The eq.

$$p(x,y) = p_X(x)\, p_Y(y)$$

must be valid for every x and y.

In general independence involves many conditions and is quite hard to check!

Example 1: (dice)

$$\frac{1}{36} = p(x,y) = p_X(x)\, p_Y(y) = \frac{1}{6} \cdot \frac{1}{6} \quad \forall x, y$$

On the contrary: example 2.

$$0.2 = p(0,0) \neq p_X(0)\, p_Y(0) = 0.3 \cdot 0.3 = 0.09$$

Not independent.

Th. if X and Y are independent and f g are functions $\mathbb{R} \to \mathbb{R}$ Then

a) $f(X)$ and $g(Y)$ are independent

b) $\mathbb{E}(f(X)\, g(Y)) = \mathbb{E}(f(X))\, \mathbb{E}(g(Y))$

In particular

$$\mathbb{E}(XY) = \mathbb{E}(X)\, \mathbb{E}(Y)$$

if $X \perp\!\!\!\perp Y$.

$$\mathbb{E}(f(X)g(Y)) = \sum_{x,y} f(x)g(y)\,p(x,y) =$$

$$\sum_{x,y} f(x)g(y)\,p_X(x)\,p_Y(y) =$$

$$\sum_{x} f(x)\,p_X(x) \sum_{y} g(y)\,p_Y(y) =$$

$$= \mathbb{E}(f(X))\,\mathbb{E}(g(Y))$$

It is also true that if

for every f and g

$$\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\,\mathbb{E}(g(Y))$$

Then

X and Y are independent.

If you have for example that

X and Y have j.p.m.f. of the

form

$$p(x,y) = e^{-(\lambda+\mu)} \frac{\lambda^x}{x!} \frac{\mu^y}{y!} =$$

$$= f(x)\, g(y)$$

$$f(x) = e^{-(\lambda+\mu)} \frac{\lambda^x}{x!} \qquad P_X(x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

$$g(y) = \frac{\mu^y}{y!} \qquad P_Y(y) = e^{-\mu} \frac{\mu^x}{x!}$$

Th: if p can be written as

$$p(x,y) = f(x)\, g(y)$$

Then X and Y are independent.